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QUANTUM DISCORD AND ENTANGLEMENT FOR TWO ATOMS IN A KERR MEDIUM AND STARK SHIFT

M. Youssef, N. Zidan*, M. Alruwaili, Y. Alshammari

Department of Mathematics, College of Science, Jouf University, Sakaka, Saudi Arabia

Abstract. This study analyzes the effect of the Kerr medium and Stark shift on the entanglement between two two-level atoms and a single-mode cavity in a quantum model. The findings demonstrate that the entanglement between the atoms varies periodically over time, with the periods affected by the Kerr medium and Stark shift. The Kerr medium and Stark shift hinder atom disentanglement, resulting in a prolonged stationary entangled state. The Stark shift can also enhance atom-atom entanglement for partially entangled states. These findings reveal that the Kerr medium and Stark shift can effectively control the entanglement between two atoms. Moreover, weak measurement strength impacts quantum particles' entanglement, which decreases over time. Quantum discord is more affected by weak measurement strengths than entanglement, decreasing until a critical point and increasing with a further rise in weak measurement strengths.

Keywords: *Kerr medium, Stark shift, Quantum discord, entanglement.*

****Corresponding Author:*** *Nour Zidan, Department of Mathematics, College of Science, Jouf University, Sakaka, Saudi Arabia, e-mail:* nazidane@ju.edu.sa

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1. Introduction

Entanglement is a quantum phenomenon that underpins various applications such as quantum teleportation, secure communication and quantum computing. It is not explicable by classical physics (Amico *et al*., 2008; Bouwmeester *et al*., 1997; Zidan *et al*., 2023; Zidan, 2014; Jennewein *et al*., 1999; Tomamichel *et al*., 2012; Sorensen & Molmer, 2000; Bakry *et al*., 2018; Redwan *et al*., 2019, 2018; Abdel-Hameed *et al*., 2017; Abd-Rabbou *et al*., 2019). Because of the decoherence and dissipation characteristics of the associated environments, the initially encoded entanglement is quickly dissipated (Czerwinski, 2022). Many researchers have investigated the techniques for efficiently preserving the entanglement in quantum systems and engineering the environment (Duan *et al*., 2000; Yu & Eberly, 2006). Bakry and Zidan (2020) conducted a study to examine how coupling strength, photon number and photon states affect the minimum uncertainty and the survival of entanglement in a quantum system. Additionally, they extensively explored how entanglement between atoms and fields can be maintained. With particular attention to multi-photon transitions, Stark shift effects, multi-mode fields, intensity-dependent coupling strengths, Kerr-medium and

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other related phenomena (Zidan *et al*., 2002; El-Shahat *et al*., 2003; Obada *et al*., 2003; Abdalla *et al*., 2005; Adel-Aty *et al*., 2002; Metwally *et al*., 2005; Abdel-Aty, 2002; Abdel-Wahab & Mourad, 2011; Hu *et al*., 2008; Baghshahi *et al*., 2014; Abdel-Khalek *et al*., 2018; Khalil & Abdel-Skalek, 2018; Adel-Khalek *et al*., 2011; Zidan *et al*., 2012; Zidan, 2012, 2010; Adel-Aty & Zidan, 2003; Akremi, 2019). Quantum discord is a measure of nonclassical correlation introduced independently by Henderson and Vedral (2001) and Ollivier and Zurek (2001). It is used to quantify the amount of quantum correlation that exists between two subsystems. Quantum discord is the difference between quantum mutual information and classical correlation in a bipartite system (Knill & Laflamme, 1998; Guhne & Toth, 2009). While the concept of quantum discord is relatively straightforward, calculating it is challenging because it requires minimization procedures and analytical solutions cannot be obtained in most cases. Only a limited set of two-qubit quantum states have explicit expressions for quantum discord. For more general quantum states, we do not yet know the expressions. However, analytical formulas for quantum discord have been obtained for a specific type of two-qubit states (Luo, 2008). Expressions for quantum discord have also been derived for more general quantum states, such as the *X*-states. It is important to note that quantum discord, entanglement and classical correlation are independent measures of correlation and there is no simple way to order them relative to each other (Ali *et al*., 2010; Wang *et al*., 2011; Zidan *et al*., 2023).

The paper is structured as follows: Section 2 details the physical model and its solution. Section 3 outlines the framework used to compute quantum discord and entanglement. Section 4 analyzes the impact of Stark shift and Kerr-like medium on quantum discord and entanglement. Finally, Section 5 presents our concluding remarks based on our findings.

2. Model with solution

Consider two non-interacting two-level atoms (A and B) resonating with a single-mode cavity through a nonlinear Kerr-like medium that can be described by a Hamiltonian (Puri & Bullough, 1988; Nasreen & Razmi, 1993; Xi-Cheng *et al*., 2010):

$$\mathrm{H_{int}} = \sum_{j=A,B}\left[a^{\dagger}a\left(\beta_2\sigma_j^{+}\sigma_j^{-} + \beta_1\sigma_j^{-}\sigma_j^{+}\right) + \lambda\left(a^{\dagger 2}\sigma_j^{-} + a^2\sigma_j^{+}\right)\right] + \chi a^{\dagger 2}a^2. \quad (1)$$

The expression involves generation and annihilation operators ($a^{\dagger}$(a)), Stark shift parameters (β_2 and β_1), the coupling between atoms and cavity (λ) and a Kerr-like medium (χ). The raising and lowering operators are $\sigma_j^{+} = |e\rangle_{jj}\langle g|$ and $\sigma_j^{-} = |g\rangle_{jj}\langle e|$, where ($|e\rangle$ and $|g\rangle$) represent excited and ground states, respectively.

Let the two atoms are initially in Bell's state, $|\Psi_{atom}(0)\rangle = \frac{1}{\sqrt{2}}\left(|eg\rangle + |ge\rangle\right)$, with $|eg\rangle$ and $|ge\rangle$representing their ground and excited states. The cavity field is initially in Fock state $|1\rangle$. Thus, the initial state reads:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|eg\rangle + |ge\rangle\right) \otimes |1\rangle. \quad (2)$$

The wave function for the current physical model changes with time and can be expressed as follows:

$$|\Psi(t)\rangle = \mu_1|eg1\rangle + \mu_2|ge1\rangle + \mu_3|gg3\rangle. \quad (3)$$

Use the Schrödinger equation:

$$i\frac{d|\Psi(t)\rangle}{dt} = H_{int}|\Psi(t)\rangle.$$

One can obtain the equations of amplitudes:

$$\left.\begin{aligned} i\frac{d\mu_1}{dt} &= (\beta_1+\beta_2)\mu_1 + \sqrt{6}\lambda\mu_3, \\ i\frac{d\mu_2}{dt} &= (\beta_1+\beta_2)\mu_2 + \sqrt{6}\lambda\mu_3, \\ i\frac{d\mu_3}{dt} &= (6\beta_1+6\chi_1)\mu_3 + \sqrt{6}\lambda\mu_1 + \sqrt{6}\lambda\mu_{2.} \end{aligned}\right\} \quad (5)$$

Then the probability amplitudes can be obtained as:

$$\left.\begin{aligned} \mu_1 = \mu_2 &= \frac{1}{\sqrt{2}\eta} exp\left[-\frac{i\lambda t\kappa}{2}\right]\left(\eta\cos\left[\frac{\lambda t\eta}{2}\right] + i\sqrt{\eta^2-48}\sin\left[\frac{\lambda t\eta}{2}\right]\right) \\ \mu_3 &= \frac{-4\sqrt{3}i}{\eta} exp\left[-\frac{i\lambda t\kappa}{2}\right]\sin\left[\frac{\lambda t\eta}{2}\right] \end{aligned}\right\}, \quad (6)$$

where $\eta = \sqrt{(5\gamma_1 - \gamma_2 + 6\chi_1)^2 + 48}, \kappa = 6\chi_1 + \gamma_2 + 7\gamma_1$ and $\gamma_1 = \beta_1/\lambda,\ \gamma_2 = \beta_2/\lambda,\ \chi_1 = \chi/\lambda$.

Using the field parameters and taking a trace of the state given by Eq. (3), we get the statistical ensemble state of the atom-atom system $\rho_{AB}(t)$. In standard basis $|ee\rangle, |ge\rangle, |eg\rangle$ and $|gg\rangle$, $\rho_{AB}(t)$ takes the form:

$$\rho_{AB}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}, \quad (7)$$

where

$$\rho_{22} = |\mu_1|^2, \qquad \rho_{33} = |\mu_2|^2, \qquad \rho_{44} = |\mu_3|^2,\ \ \rho_{23} = \mu_1\mu_2^*, \qquad \rho_{32}^* = \rho_{23}\,. \quad (8)$$

3. Quantum discord, entanglement and weak measurement reversal

3.1. Quantum discord (QD)

Quantum discord represents the difference between quantum mutual information and classical correlation (Henderson & Vedral, 2001) where quantum mutual information is the ultimate measure of the total correlation of a two-qubit quantum system:

$$\mathrm{TC}(\rho_{AB}) = \mathrm{S}(\rho_A) + \mathrm{S}(\rho_B) - \mathrm{S}(\rho_{AB}), \quad (9)$$

where $\rho_{A(B)}$ and ρ_{AB} are reduced and system densities respectively and their von Neumann entropy $S(\rho) = -Tr(\rho \log_2 \rho)$. Quantum discord quantifies the quantum correlation between A and B. It is the difference between total and classical correlation:

$$\mathrm{QD}(\rho_{\mathrm{AB}}) = \mathrm{TC}(\rho_{\mathrm{AB}}) - \mathrm{CC}(\rho_{\mathrm{AB}}). \tag{10}$$

Here, the classical correlation $\mathrm{CC}(\rho_{\mathrm{AB}})$ quantifies the maximum information that one can obtain from the composite system by measuring one of the subsystems:

$$\mathrm{CC}(\rho_{\mathrm{AB}}) = \max_{\{B_k\}}[\mathrm{S}(\rho_{\mathrm{A}}) - \mathrm{S}(\rho_{\mathrm{AB}}|\{B_k\})], \tag{11}$$

where $\{B_k\}$ represents a complete set of projective measurements that are performed locally on subsystem B and $\mathrm{S}(\rho_{\mathrm{AB}}|\{B_k\}) = \sum_k p_k S(\rho_k)$ is the quantum conditional entropy with $[(I\otimes B_k)\rho_{\mathrm{AB}}(I\otimes B_k)]/p_k$ and $p_k = Tr[(I\otimes B_k)\rho_{\mathrm{AB}}(I\otimes B_k)]$. Finally, the quantum discord can be obtained as:

$$\mathrm{QD}(\rho_{\mathrm{AB}}) = \mathrm{S}(\rho_{\mathrm{B}}) - \mathrm{S}(\rho_{\mathrm{AB}}) + \min_{\{B_k\}} \sum_k p_k S(\rho_k). \tag{12}$$

However, for the density matrix in *X*-state

$$\rho_{AB}(t) = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{13}$$

Quantum discord (QD) is expressed as (Wang *et al*., 2011):

$$QD = min[QD_1, QD_2], \tag{14}$$

with

$$QD_j = \Gamma(\rho_{11} + \rho_{33}) + \sum_{\mathrm{i}=1}^{4} \lambda_{\mathrm{i}} \log_2 \lambda_{\mathrm{i}} + \mathrm{R_j}$$

and

$$R_1 = -\Gamma(\rho_{11} + \rho_{33}) - \sum_{\mathrm{i}=1}^{4} \rho_{\mathrm{ii}} \log_2 \rho_{\mathrm{ii}}$$

$$R_2 = \Gamma\left(\frac{1 + \sqrt{(\rho_{11} + \rho_{22} - \rho_{33} - \rho_{44})^2 + 4(|\rho_{23}|^2 + |\rho_{14}|^2)}}{2}\right),$$

where λ_{i} are eigenvalues of ρ_{AB} with $\Gamma(\mathrm{O}) = -\mathrm{O}\ \log_2 \mathrm{O} - (1 - \mathrm{O}) \log_2(1 - \mathrm{O})$.

For the density matrix of the system under consideration given by Eq. (7), the quantum discord can be obtained as:

$$QD = min[QD_1, QD_2], \tag{15}$$

with

$$QD_1 = \sum_{i=1}^{3} \lambda_i \log_2 \lambda_i - \rho_{22} \log_2 \rho_{22} - \rho_{33} \log_2 \rho_{33} - \rho_{44} \log_2 \rho_{44},$$

$$QD_2 = \Gamma(\rho_{33}) + \sum_{i=1}^{3} \lambda_i \log_2 \lambda_i - \Gamma\left(\frac{1 + \sqrt{(\rho_{22} - \rho_{33} - \rho_{44})^2 + 4|\rho_{23}|^2}}{2}\right)$$

and $\lambda_1 = \rho_{44}$, $\lambda_{2,3} = \frac{1}{2}\left(\rho_{22} + \rho_{33} \pm \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{23}|^2}\right)$.

3.2. Entanglement

For the entanglement estimation, we use concurrence is defined as (Hill & Wootters, 1997; Wootters, 1998):

$$C = max\left(0, \sqrt{\mathcal{L}_1} - \sqrt{\mathcal{L}_2} - \sqrt{\mathcal{L}_3} - \sqrt{\mathcal{L}_4}\right), \tag{16}$$

where $\mathcal{L}_i (i = 1,2,3,4)$ are the eigenvalues of the matrix $\tilde{\rho} = \rho_{AB}(\sigma_y \otimes \sigma_y)\rho_{AB}^{\dagger}(\sigma_y \otimes \sigma_y)$ in decreasing order. Using Eq. (7), the concurrence can be put into the form:

$$\mathrm{C} = 2\max[0, |\rho_{23}|], \tag{17}$$

where for C = 0, 1, the state will be assumed completely separable and maximally entangled, respectively.

3.3. Weak measurement reversal

We discuss the measurement of quantum discord and entanglement with weak measurement reversal (Sun *et al*., 2010; Huang, 2018). This measurement has a strength of $p(0 \leq p < 1)$.

$$\mathrm{M_w} = \begin{pmatrix} \sqrt{1-\mathrm{p}} & 0 \\ 0 & 1 \end{pmatrix}. \tag{18}$$

After applying a weak measurement reversal to qubit A, the system's post-selection state is obtained.

$$\varrho = \frac{(\mathrm{M_w} \otimes \mathrm{I})\rho(\mathrm{M_w} \otimes \mathrm{I})^{\dagger}}{\mathrm{Tr}[(\mathrm{M_w} \otimes \mathrm{I})\rho(\mathrm{M_w} \otimes \mathrm{I})^{\dagger}]}, \tag{19}$$

with probability $Tr[(M_w \otimes I)\rho(M_w \otimes I)^{\dagger}]$. After performing a weak measurement reversal on qubit A, the system's post-selection state is obtained:

$$\varrho = \frac{1}{1-p+p(\rho_{33}+\rho_{44})}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (1-p)\,\rho_{22} & \sqrt{1-p}\;\rho_{23} & 0 \\ 0 & \sqrt{1-p}\,\rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{20}$$

Using the above computational equation and methods, we can calculate the quantum discord and entanglement of the state after post-selection.

4. Numerical discussion

Here, we will share the results of our study on entanglement. We used two methods, concurrence (C) and quantum discord (QD), to determine the entanglement levels in our current configuration. Our main goal was to find a way to preserve entanglement in two-level atoms for a more extended period. To achieve this, we explored the unique properties of the Stark shift and Kerr medium.

4.1. Concurrence (C)

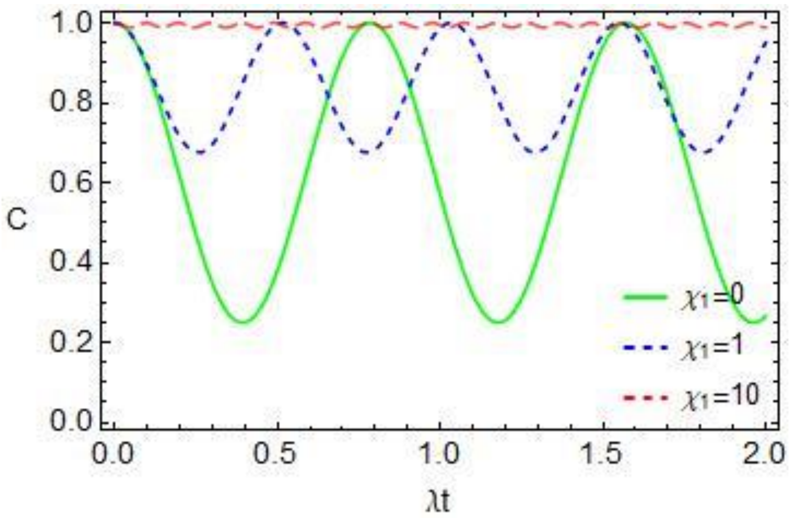

Figure 1. The concurrence (C) against the scaled time λt under different values of Kerr medium (χ_1), where $\beta_1 = \beta_2 = 1.0$

In Figure 1, we plot the concurrence (C) against the scaled time λt under different values of Kerr medium (χ_1). Figure 1 shows that the entanglement between the atoms depends on the value of χ_1. The results show that the dynamic Kerr medium significantly impacts the entanglement's periodicity. The figure also shows that the minimum value of the concurrence (C) increases with an increase in χ_1, which leads to long-lived entanglement well-supported by the blue-dashed lines. In other words, when the value of χ_1 increases, the entanglement between the atoms becomes more robust and lasts longer. By increasing χ_1, we can observe a steady state of the entanglement, apparent from the results obtained from the red-dashed line in Figure 1. This means that when the coupling strength is increased, the entanglement between the atoms reaches a steady state that persists over time.

Figure 2 presents a plot showing the concurrences C as a function of λt, illustrating how the Stark shift affects the entanglement between two atoms. It is important to note that the atoms remain entangled even with the Stark shift. When we compare Figure 2 to Figure 1, we can see some differences in how the entanglement behaves. The concurrence evolution is affected not only by the Stark shift but is also periodic, indicating that the entanglement between the two atoms oscillates over time, reflecting the system's dynamic nature. It is also worth mentioning that if the atomic state is initially partially entangled, the entanglement between the two atoms can be significantly improved due to the presence of Stark shift. This highlights the crucial role of the dynamic Stark shift in maintaining the long-lived entanglement state between two-level atoms.

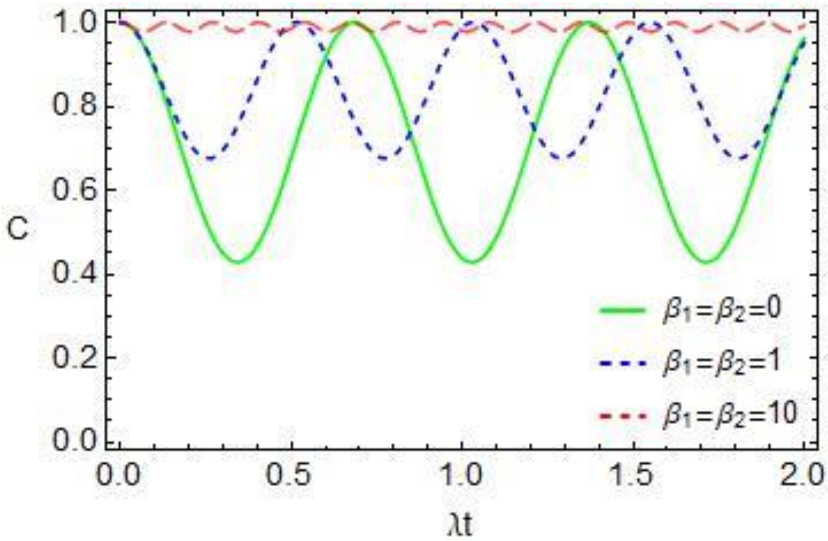

Figure 2. The concurrence (C) against the scaled time λt under the different values of Stark shift ($\beta_1 = \beta_2$), where $\chi_1 = 1.0$

4.2. Quantum discord (QD)

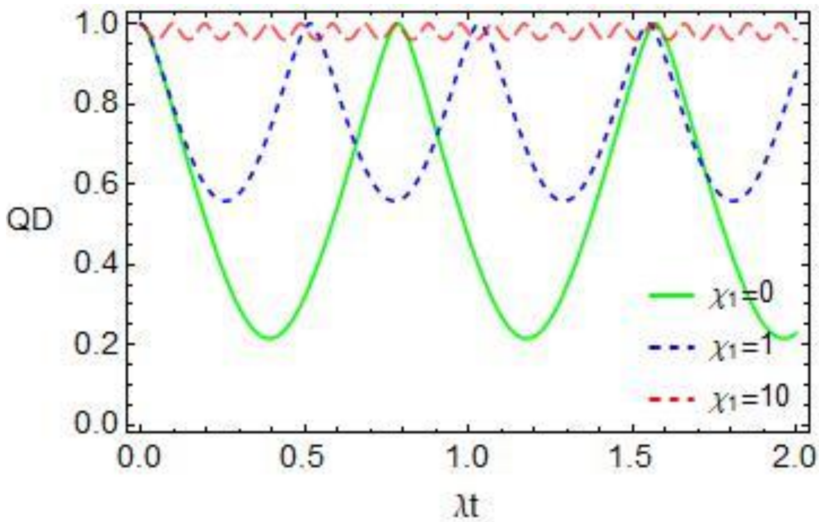

Figure 3. The quantum discord (QD) against the scaled time λt under the different values of Kerr medium (χ_1), where $\beta_1 = \beta_2 = 1.0$

Figure (3) demonstrates the behavior of quantum discord as a function of the scaled time λt under the different values of Kerr medium (χ_1). The lines in Figure (3) show that the quantum discord evolves periodically and the presence of a Kerr medium affects the periods. The minimum value of quantum discord increases with an increase in χ_1, leading to long-lived quantum discord, as indicated by the blue-dashed lines. If we further increase the value of the parameter χ_1 to 10 or more, corresponding to a more robust coupling strength between the atomic ground and intermediate level, we observe a steady state of quantum discord. This implies that the two atoms can remain stationary and entangled. The steady state of quantum discord arises due to the balance between the system-environment interaction and the internal dynamics of the atoms. Therefore, the value of χ_1 plays a crucial role in determining the nature of the quantum correlations between the two atoms.

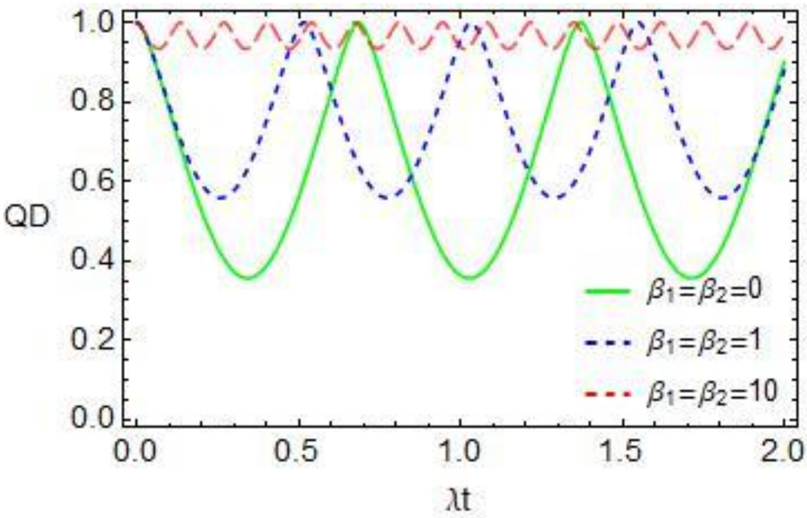

Figure 4. The quantum discord (QD) against the scaled time λt under the different values of Stark shift ($\beta_1 = \beta_2$), where $\chi_1 = 1.0$

Figure 4 illustrates the impact of the Stark shift on the quantum discord between two atoms, as shown by the quantum discord (QD) versus λt. Despite the Stark shift, the atoms remain entangled. Comparing Figure 4 with Figure 3, we can observe differences in the quantum discord dynamics. However, we notice that the time evolution of quantum discord and the effect of the Stark shift on entanglement are similar. The quantum discord follows a periodic evolution. The stark shift can significantly enhance the quantum discord between the two atoms for initially partially entangled atomic states. This Stark shift allows the two-level atoms to remain in a long-lived entangled.

4.3. Weak measurement reversal

In Figure 5, we will delve deeper into the impact of weak measurement strength, represented by the variable p, on the concurrence. The concurrence is plotted as a function of scaled time and p to reveal how the change in p affects the entanglement between two quantum particles. Observing Figure 5 notes that when p = 0, the concurrence fluctuates periodically and its value is at the highest possible level. This suggests that the entanglement between the two particles is at its peak when no weak measurement strength is applied. As p increases, the value of concurrence decreases and this decrease continues until the concurrence reaches its lowest point of entanglement when p = 1. This means the entanglement between the two particles is weakest when the maximum weak measurement strength is applied. However, it is essential to note that the periodic fluctuation behavior of concurrence remains even now. This suggests that the entanglement between the two particles still exists, but it is at its weakest point. Therefore, Figure 5 shows that the strength of a weak measurement significantly impacts the entanglement between two quantum particles. As the strength of weak measurement increases, the entanglement between the two particles decreases with time, but it remains even at its weakest point.

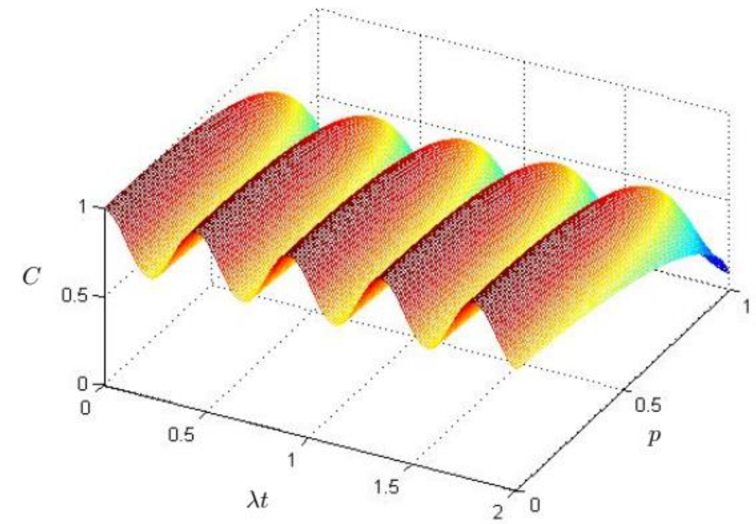

Figure 5. The Concurrence (C) against the scaled time λt and weak measurement strengths p, with $\beta_1 = \beta_2 = 1.0$ and $\chi_1 = 1.0$

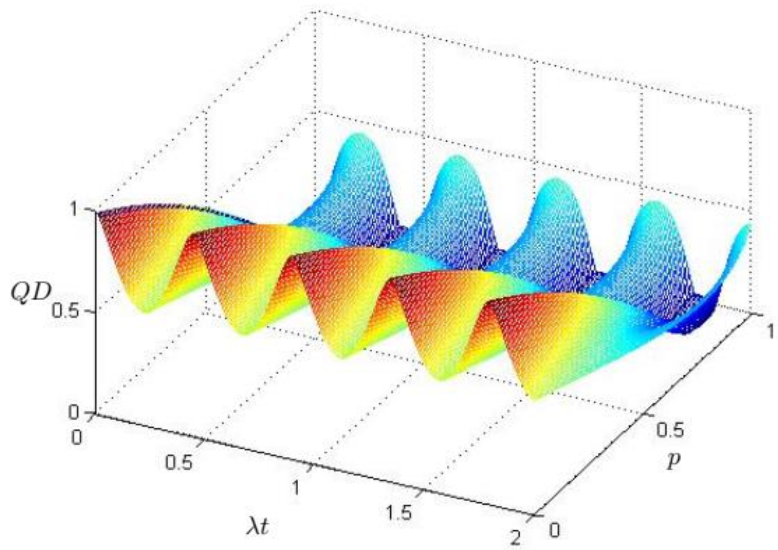

Figure 6. The quantum discord (QD) against the scaled time λt and weak measurement strengths p, with $\beta_1 = \beta_2 = 1.0$ and $\chi_1 = 1.0$

A detailed analysis of the effect of weak measurement strengths p on Quantum Discord (QD) is presented in Figure 6. The plot depicts the fluctuation of QD concerning scaled time λt and weak measurement strengths p. The results show that QD fluctuates periodically as t increases. Moreover, it is found that the impact of weak measurement strengths p on QD is more substantial than the effect of entanglement. During the analysis, it is observed that QD decreases with an increase in p until the value of p reaches a critical point. After this point, QD increases with a further rise in p. However, the most significant possible value of QD remains when $p = 0$. Hence, it can be concluded that increasing p does not weaken the value of QD. According to the findings, weak measurement can potentially affect the quantum state. Based on this, it is suggested that further research be conducted to explore the impact of weak measurement on other properties of quantum systems. These results contribute to better comprehending quantum mechanics and its applications in diverse fields.

5. Conclusion

Our study analyzed a quantum model of two two-level atoms and a single-mode cavity. The atoms interacted with the cavity field through a two-photon process but not with each other. We focused on the entanglement dynamics between the atoms, assuming they were initially entangled while the cavity was in a Fock state $|1\rangle$. Our findings reveal that the entanglement oscillated periodically due to the cavity field's interaction and that the field's parameters affected the entanglement dynamics.

Our investigation explored how the dynamic Kerr medium and Stark shift can affect quantum discord and entanglement. We also investigated the possibility of using them to control and protect these phenomena. The dynamic Kerr medium and Stark shift can impact the phase evolution of the atoms and the cavity field.

Our results showed that the quantum discord and concurrence between the two atoms followed a periodic pattern influenced by the dynamic Kerr medium and Stark shift. Additionally, we found that the Kerr medium and Stark shift could result in long-lived entanglement between the two atoms. The two atoms could remain stationary and entangled even with high Kerr medium and Stark shift parameter values.

In addition, we explored the impact of the Kerr medium and Stark shift on the partially entangled initial atomic state. The Kerr medium and Stark shift can significantly enhance the quantum discord and entanglement between the two atoms. The maximum value of the enhanced quantum discord and entanglement can reach 1. The non-linear coupling in two-photon processes may explain the observed behaviors.

Our study demonstrates that the dynamic Kerr medium and Stark shift influence quantum discord and entanglement dynamics. Therefore, these factors can be utilized to control the quantum discord and entanglement between two atoms. Our findings may have implications for quantum information processing and quantum communication.

It has been demonstrated that the strength of a weak measurement significantly impacts the entanglement between two quantum particles. As the strength of a weak measurement increases, the entanglement between the two particles decreases over time, but it remains even at its weakest point.

Upon analyzing the data, it was observed that QD (Quantum Discord) decreases as the value of weak measurement strengths increases until it reaches a critical point. Beyond this point, QD increases with a further rise in weak measurement strengths.

However, it can be concluded that increasing the value of weak measurement strengths does not weaken the value of QD.

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